Problem 1: Sum of Multiples of 3 or 5

Problem

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23. Find the sum of all the multiples of 3 or 5 below the provided input number.

Solution

All the common Multiples of 3 and 5 will also be the multiple of 15.

Example:-

Multiples of 3:-	3	6	9	12	15	18	21	24	27	30	⋯
Multiples of 5:-	5	10	15	20	25	30	35	⋯			
Multiples of 15:-	15	30	45	⋯							

Hence if we add all the multiples of 3 and 5 till the given number, multiples of 15 (i.e. the intersection will be added twice)

Let S_3, S_5, S_{15} be the sum of all the multiples of 3, 5 and 15 up to the given number (i.e. the input given by user) respectively.

∴ By applying the principle of inclusion and exclusion we can directly calculate output:-

$$\boldsymbol{Output = S_3 + S_5 - S_{15}}$$

Let t_3, t_5, t_{15} be the number of multiples of 3, 5, 15 below the given $input\ (i)$ respectively.

$\therefore \boldsymbol{t_3 = \left\lfloor \frac{i}{3} \right\rfloor}$	$\therefore \boldsymbol{t_5 = \left\lfloor \frac{i}{5} \right\rfloor}$	$\therefore \boldsymbol{t_{15} = \left\lfloor \frac{i}{15} \right\rfloor}$

where $\lfloor \cdot \rfloor$ represents Greatest Integer Function.

$$\therefore S_3 = 3 + 6 + 9 + \cdots + 3t_3 = \sum_{r=1}^{t_3} 3r = 3\sum_{r=1}^{t_3} r = 3 \cdot \frac{t_3(t_3+1)}{2} = \frac{3}{2} \cdot t_3(t_3+1)$$

Similarly,

$$S_5 = 5 + 10 + 15 + \cdots + 5t_5 = \frac{5}{2} \cdot t_5(t_5+1)$$

$$S_{15} = 15 + 30 + 45 + \cdots + 15t_{15} = \frac{15}{2} \cdot t_{15}(t_{15}+1)$$

Conclusion

$$\boldsymbol{Output = \frac{3}{2} \cdot \left\lfloor \frac{i}{3} \right\rfloor \left(\left\lfloor \frac{i}{3} \right\rfloor + 1 \right) + \frac{5}{2} \cdot \left\lfloor \frac{i}{5} \right\rfloor \left(\left\lfloor \frac{i}{5} \right\rfloor + 1 \right) - \frac{15}{2} \cdot \left\lfloor \frac{i}{15} \right\rfloor \left(\left\lfloor \frac{i}{15} \right\rfloor + 1 \right)}$$

Where $\lfloor \cdot \rfloor$ represents Greatest Integer Function.

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